Event Based Agreement Protocols for Multi-agent Networks

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Why consensus??

Modern

Omni-present, omni-potent controller that decides what to do. . .

Post-modern

Agreement from different initial points of view. . . every agent influences the final view. . .

Contents

Main ideas and contributions

- Model the network using graph theory
- Design an event-based control law to reduce communication and controller's updating frequency, while achieving consensus
- Extend the results for switching networks

- Sample-data event detector
- Event-based consensus algorithms for switching topologies using sample-data event detection

Graph theory

Consider a network of N agents. The communication among agents is modeled by an undirected (bidirectional) graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$:

- $\mathcal{V} := \{1, \cdots, N\}$ is the node set of the graph
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set of the graph
- $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}, i \neq j\}$ represents the neighbors of i and $|\mathcal{N}_i|$ its cardinality
- A path is a (finite) sequence of edges
- $\mathcal{G}$ is connected if there is a path between any pair of nodes

Graph theory – matrix representations

- $\mathscr{A}(\mathscr{G})$ is the adjacency matrix

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathscr{E} \\ 0, & \text{if } (i,j) \notin \mathscr{E} \end{cases}$$

- $\mathscr{D}(\mathscr{G})$ is the degree matrix

$$\text{diag}\{|\mathscr{N}_1|, |\mathscr{N}_2|, \ldots, |\mathscr{N}_N|\} \tag{1}$$

- $\mathscr{L}(\mathscr{G})$ is the Laplacian matrix

$$\mathscr{L}(\mathscr{G}) = \mathscr{D}(\mathscr{G}) - \mathscr{A}(\mathscr{G}) \tag{2}$$

The following claims hold for $\mathscr{L}(\mathscr{G})$ when $\mathscr{G}$ is connected:

$$\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N \tag{3}$$

$$\mathbf{1}^T \mathscr{L} = 0 \tag{4}$$

Formulation and event condition

Consider a network of N agents $x_i \in \mathbb{R}$ modeled as a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with dynamics

$$\dot{x}_i = u_i(t) \tag{5}$$

The asymptotic event based consensus problem is solved if one can find an event based protocol such that

$$||x_i(t) - x_j(t)||_2 \to 0, \text{ as } t \to \infty \; \forall i, j \in \mathcal{V} \tag{6}$$

The proposed sample-data event condition for agent i is of the form

$$\left\| e_i\left(t_k^i + lh\right)\right\|_2^2 \leq \sigma_i \left\| z_i\left(t_k^i + lh\right)\right\|_2^2, \; l = 1, 2, \ldots \tag{7}$$

where: $\sigma_i \in \mathbb{R}_{>0}$, t_k^i is kth event instant for i, h is the sampling period.

$$e_i\left(t_k^i + lh\right) = x_i\left(t_k^i\right) - x_i\left(t_k^i + lh\right) \tag{8}$$

and

$$z_i\left(t_k^i + lh\right) = \sum_{j \in \mathcal{N}_i} x_i\left(t_k^i + lh\right) - x_j\left(t_k^i + lh\right) \tag{9}$$

Formulation and event condition

Remark

- *At each sampling instant, each agent broadcasts its state and receives information*
- *If (7) holds, nothing happens*
- *if (7) does not hold, i will update its control value and "notify" neighbors to update their control values*

Remark

The control law is piecewise constant between the sampling instants $\{h, 2h, 3h, \dots\}$

In the following consider:

$$\hat{x}_i(t) := x_i\left(t_k^i\right), \quad \text{for } t_k^i \leq t < t_{k+1}^i \tag{10}$$

Consensus controller

Theorem

Consider the system $\dot{x}_i = u_i$ *over a connected communication graph with the protocol*

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} \left(\hat{x}_i(t) - \hat{x}_j(t) \right) \tag{11}$$

driven by the event condition

$$\left\| e_i \left(t_k^i + lh \right) \right\|_2^2 \leq \sigma_i \left\| z_i \left(t_k^i + lh \right) \right\|_2^2 \tag{12}$$

Then all agents are asymptotically converging to their initial average if

$$0 < h \leq \frac{1}{2\lambda_N} \quad \textit{and} \quad 0 < \sigma_{\max} < \frac{1}{\lambda_N^2} \tag{13}$$

Consensus controller – some remarks

To choose h and σ_i, global knowledge is needed. Can be fixed by noting that:

$$\lambda_N \leq 2d_{\max} \leq 2(N-1) \tag{14}$$

then

$$0 < h \leq \frac{1}{4(N-1)} \quad \text{and} \quad 0 < \sigma_{\max} < \frac{1}{4(N-1)^2} \tag{15}$$

Interesting to notice:

1. It is more realistic to approximate using small h and σ_i
2. Small σ_i leads to faster convergence but higher frequency control updates
3. Larger h reduces communication load

Consensus controller – proof

The closed loop system is given by:

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} \left(\hat{x}_i(t) - \hat{x}_j(t) \right) \Rightarrow \dot{x} = -\mathcal{L}\hat{x}(t) \tag{16}$$

Denote the average as $\bar{x}$ and note that:

$$\dot{\bar{x}}(t) = \frac{1}{N}\sum \dot{x}_i(t) = \frac{1}{N}\mathbf{1}^T\dot{x}(t) = -\frac{1}{N}\mathbf{1}^T\mathcal{L}\hat{x}(t) = 0 \tag{17}$$

we then define the disagreement vector as:

$$\delta(t) = x(t) - \bar{x}\mathbf{1} \tag{18}$$

Using the definition of $e\left(t_k^i + lh\right)$, the dynamics for $t \in [kh, (k+1)h)$ can be written as:

$$\dot{x}(t) = -\mathcal{L}x(kh) - \mathcal{L}e(kh) \tag{19}$$

Consensus controller – proof

Consider the following Lyapunov function:

$$V(x) = \frac{1}{2} x^T(x) x(t) \tag{20}$$

and note that

$$V(x) = \frac{N}{2} \sum \frac{1}{N} x_i^2 \geq \frac{N}{2} \left(\sum \frac{1}{N} x_i \right)^2 = \frac{N}{2} \bar{x}^2 = V(\bar{x}\mathbf{1}) \tag{21}$$

after some manipulations, and assuming $0 < h \leq \frac{1}{2\lambda_N}$ and $0 < \sigma_{\max} < \frac{1}{\lambda_N^2}$ we get:

$$\dot{V}(t) \leq -\frac{1}{2} \left(1 - \lambda_N^2 \sigma_{\max} \right) x^T(kh) \mathscr{L} x(kh) \leq 0 \tag{22}$$

Invoking the invariance principle yields that the state converges to the set:

$$\left\{ x \in \mathbb{R}^N \mid \dot{V}(t) = 0 \right\} = \text{span}\{\mathbf{1}\} \tag{23}$$

Consensus controller

Theorem

Consider the system $\dot{x}_i = u_i$ switches over a number of connected communication graphs with the protocol

$$u_i(t) = -\sum_{j\in\mathcal{N}_i(\mathcal{G})} \left(\hat{x}_i(t) - \hat{x}_j(t)\right) \tag{24}$$

driven by the event condition

$$\left\| e_i\left(t_k^i + lh\right)\right\|_2^2 \leq \sigma_i \left\| z_i\left(t_k^i + lh\right)\right\|_2^2 \tag{25}$$

Then all agents are asymptotically converging to their initial average if

$$0 < \sigma_{\max} < \frac{1}{\lambda_{\max}^2} \quad \text{and} \quad 0 < h \leq \frac{1}{2\lambda_{\max}} \tag{26}$$

where $\lambda_{\max} = \max\left\{\lambda_N(\mathcal{G}), \mathcal{G} \in \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}\right\}$

Consensus controller – some remarks

1. Switching signal $s(t) : [0, \infty) \to J := \{1, \dots, m\}$
2. Topology can change at any time instant
3. Several switchings may happen between sampling instants
4. The same Lyapunov function $V(t) = \frac{1}{2} x^T(t) x(t)$ gives convergence

Some open issues:

- Switching between connected graphs. . .
- Conservative bound??

Preliminaries

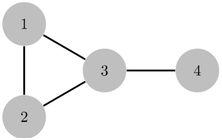
Multi-agent Networks with Fixed Topology

Multi-agent Networks with Switching Topology

Numerical Simulations

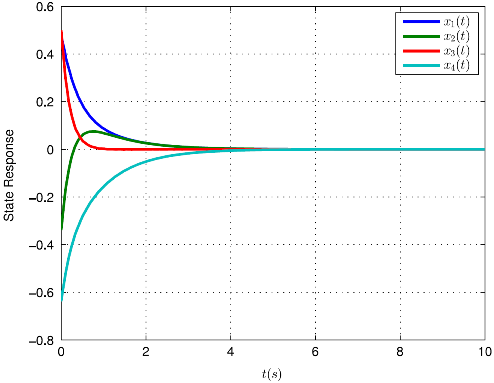
Concluding Remarks

Fixed topology



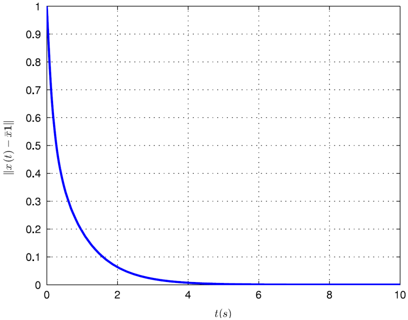
$x(0) = [0.4773; -0.3392; 0.5; -0.6381]$

- $\bar{x} = 0$
- $\lambda_N = 4$
- $\sigma_{\max} < 0.0625,\ h \leq 0.125$
- $\sigma_1 = \sigma_2 = 0.033,\ \sigma_3 = 0.02,$
  $\sigma_4 = 0.06$
- $h = 0.0002$

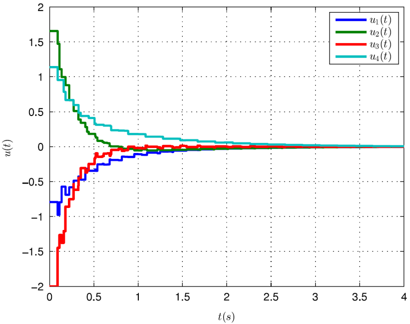


State evolution for each agent

Fixed topology



Evolution of agreement error



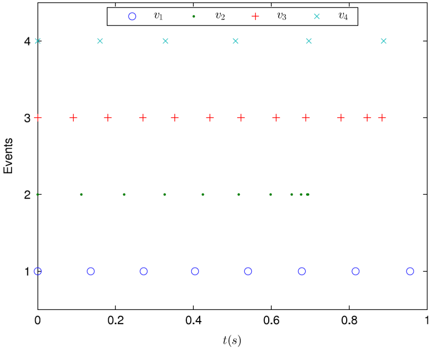
Evolution of control inputs

Fixed topology

Event intervals summary

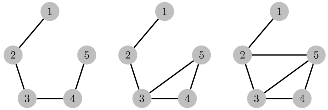
Agent	1	2	3	4
Events	66	69	75	52
Min	0.132	0.002	0.002	0.160
Mean	0.152	0.145	0.131	0.196
Max	0.156	0.346	0.614	0.198

- Event times lower bounded by sampling period
- No events after $t \approx 4$

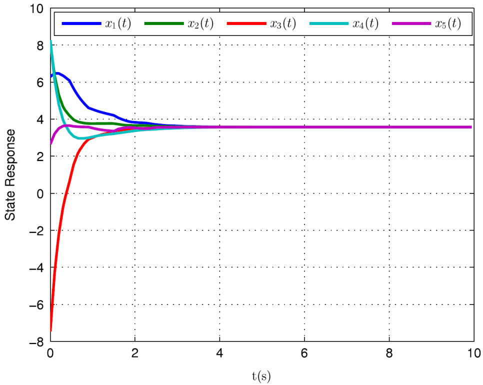


Event times for each agent

Switching topology

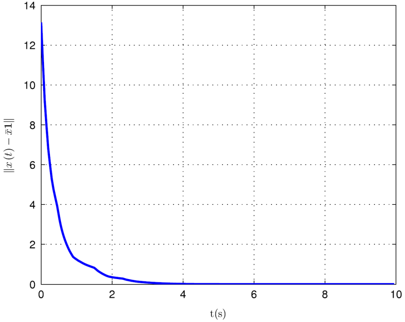


- $x_i(0) \sim U[-10, 10]$
- Initial topology $\mathscr{G}_1$
- Dwell time $t_d \sim U[0.1, 0.5]$
- Topology index $J \sim U\{1, 2, 3\}$
- $\sigma_i = 0.02, \ h = 0.05$



State evolution for each agent

Switching topology



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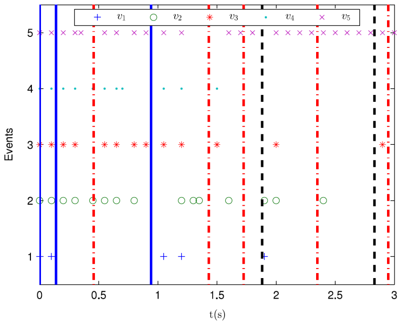

Event times for each agent

Paper conclusions:

- A new Lyapunov function was introduced (?)
- Sample-data event detectors were designed to achieve average consensus
- Based on the results for fixed topologies, an event based consensus algorithm was proposed for switching networks

Open issues:

- Common sampling time
- Only connected graphs
- Conservativeness
- Directed networks?

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