Sufficient Conditions For Exogenous Input Estimation In Nonlinear Systems

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***Abstract*—A large class of nonlinear plants subject to unknown but bounded exogenous inputs (disturbance inputs and sensor noise) is considered. A convex programming method for the design of state observers for these plants is proposed. These observers can be used to estimate the exogenous inputs to some specified degree of accuracy. Additionally, conditions guaranteeing estimation of the disturbance inputs with arbitrary accuracy are presented. The effectiveness of the proposed observers is illustrated with a numerical example.**

I. Introduction

In noisy environments, it is imperative to design observers for dynamical systems which perform at pre-specified performance levels. From a practical viewpoint, it is desirable to estimate the state while simultaneously reconstructing the unknown (exogenous) inputs. For example, in secure networks, unknown input reconstruction is useful for the mitigation of attack signals. Another example could be the reconstruction of unmodeled physiological inputs in biomedical systems.

Unknown input reconstruction for linear systems has been studied using linear observers in [1]–[3], and using sliding mode observers in [4]–[6]. Other unknown input observer architecture for globally Lipschitz nonlinear systems are proposed in, for example, [7]–[13]. Extensions to monotone nonlinearities and slope-restricted nonlinearities are discussed in [14]–[16].

In this paper, we present a systematic framework for the design of observers for a class of nonlinear dynamical systems in the presence of bounded disturbance inputs and bounded sensor noise. We lump the disturbance input and sensor noise into a single exogenous input. The class of nonlinearities under consideration are characterized by a set of symmetric matrices. Our **contributions** include: (i) a convex programming framework for designing observers for nonlinear systems with exogenous inputs; (ii) providing performance guarantees and explicit bounds on the unknown input reconstruction error; (iii) providing conditions for unknown input reconstruction in nonlinear systems with arbitrary accuracy; and, (iv) for linear error dynamics, demonstrating that our proposed LMIs are a generalization of existing conditions for unknown input observers. Proofs omitted in this paper for space considerations are available at [17].

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II. Notation

We denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices. For any matrix P, we denote $P^\top$ as its transpose, and $\|P\|$ as the maximum singular value of P. For any vector $v \in \mathbb{R}^n$, we consider the norm $\|v\| = \sqrt{v^\top v}$. For a bounded function $v(\cdot) : \mathbb{R} \to \mathbb{R}^n$, we consider the norm $\|v(\cdot)\|_\infty = \sup_t \|v(t)\|$. For a symmetric matrix $M = M^\top$, we use the star notation to avoid rewriting symmetric terms, that is,

$$\begin{bmatrix} M_a & \star \\ M_b^\top & M_c \end{bmatrix} \equiv \begin{bmatrix} M_a & M_b \\ \star & M_c \end{bmatrix} \equiv \begin{bmatrix} M_a & M_b \\ M_b^\top & M_c \end{bmatrix}.$$

We also denote $\mathcal{D}f$ as the derivative of a differentiable function f.

III. Problem Statement and Proposed Solution

We consider a **nonlinear system** with disturbance input, measured output and measurement noise described by

$$\dot{x} = Ax + B_n f(t,y,q) + Bw + g(t,y) \tag{1a}$$
$$q = C_q x + D_{qn} f(t,y,q) + D_{qw} w \tag{1b}$$
$$y = Cx + Dw \tag{1c}$$

where $t \in \mathbb{R}$ is the **time variable**, $x(t) \in \mathbb{R}^{n_x}$ is the **state**, $y(t) \in \mathbb{R}^{n_y}$ is the **measured output** and the vector $w(t) \in \mathbb{R}^{n_w}$ models the **disturbance input** and the **measurement noise** combined into one term; we refer to it as the **exogeneous input**. This exogenous input is unknown but bounded.

The vector $f(t,y,q) \in \mathbb{R}^{n_f}$ models nonlinearities of known structure, but because this term depends on the state x (through q), it cannot be instantaneously determined from measurements. The vector $g(t,y) \in \mathbb{R}^{n_g}$ represents nonlinearities which can be calculated instantaneously from measurements. The matrices $A \in \mathbb{R}^{n_x\times n_x}, B \in \mathbb{R}^{n_x\times n_w}, B_n \in \mathbb{R}^{n_x\times n_f}$ $C \in \mathbb{R}^{n_y\times n_x}$ and $D \in \mathbb{R}^{n_y\times n_w}$ describe how the variables x, w and f enter the state and output equations of the system. The vector $q \in \mathbb{R}^{n_q}$ is a state-dependent argument of the nonlinearity f, and is characterized by the matrices $C_q \in \mathbb{R}^{n_q\times n_x}$, $D_{qn} \in \mathbb{R}^{n_q\times n_f}$ and $D_{qw} \in \mathbb{R}^{n_q\times n_w}$ as shown in (1b). The quantity q is not measured instantaneously and has to be estimated. The $D_{qw}w$ term enables us to model an exogenous input acting through the nonlinearity f.

Remark 1. If the plant has a control input or other known inputs, this can be included in the term g.

The plant trajectories are defined as continuous functions $x(\cdot) : [t_0, t_1) \to \mathbb{R}^{n_x}$, with $0 < t_1 \leq \infty$ satisfying equations (1a)–(1b).

In this paper we characterize nonlinearities via their incremental multiplier matrices.

Definition 1 (Incremental Multiplier Matrices). A symmetric matrix $M \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ is an **incremental multiplier matrix** (δMM) for f if it satisfies the following **incremental quadratic constraint** (δQC) for all $t \in \mathbb{R}$, $y \in \mathbb{R}^{n_y}$ and $q_1, q_2 \in \mathbb{R}^{n_q}$:

$$\begin{bmatrix}\Delta q\\ \Delta f\end{bmatrix}^\top M \begin{bmatrix}\Delta q\\ \Delta f\end{bmatrix} \geq 0, \tag{2}$$

where $\Delta q \triangleq q_1 - q_2$ and $\Delta f \triangleq f(t, y, q_1) - f(t, y, q_2)$.

Example 1. Consider the nonlinearity $f(t, y, q) = q^3$, which is not globally Lipschitz. The nonlinearity f satisfies the inequality $(q_1^3 - q_2^3)(q_1 - q_2) \geq 0$, for any $q_1, q_2 \in \mathbb{R}$. This can be rewritten as

$$\begin{bmatrix}q_1 - q_2\\ q_1^3 - q_2^3\end{bmatrix}^\top \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \begin{bmatrix}q_1 - q_2\\ q_1^3 - q_2^3\end{bmatrix} \geq 0\,.$$

Hence, an δMM for $f(q)$ is

$$M = \kappa \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

for any $\kappa > 0$.

Remark 2. Clearly, if a nonlinearity has a non-zero incremental multiplier matrix, it is not unique. Any positive scalar multiplier of an δMM is also an δMM.

To ensure that the implicit description of q results in a unique explicit description of q we need the following assumption on the nonlinearity f and D_{qn}.

Assumption 1. The nonlinear function f satisfies an incremental quadratic constraint. Furthermore, there exists a continuous function ψ such that for every $t \in \mathbb{R}$, $y \in \mathbb{R}^{n_y}$ and $\tilde{q} \in \mathbb{R}^{n_q}$, the equation

$$q = \tilde{q} + D_{qn} f(t, y, q)$$

has a unique solution given by $q = \psi(t, y, \tilde{q})$, that is,

$$\psi(t, y, \tilde{q}) = \tilde{q} + D_{qn} f(t, y, \psi(t, y, \tilde{q})). \tag{3}$$

Thus, in model (1), q is explicitly given by

$$q = \psi(t, y, C_q x + D_{qw} w). \tag{4}$$

The utility of characterizing nonlinearities using incremental multipliers is that we can generalize our observer design strategy for a broad class of nonlinear systems. Incremental multiplier matrices for many common nonlinearities are provided in [18], [19].

IV. Observer Design

In this section, we propose an **observer** architecture and provide conditions that guarantee observer performance in the presence of exogenous inputs.

A. *Proposed observer and error dynamics*

Our proposed observer is described by

$$\dot{\hat{x}} = A\hat{x} + B_n f(t, y, \hat{q}) + L_1(\hat{y} - y) + g(t, y) \tag{5a}$$
$$\hat{q} = C_q \hat{x} + D_{qn} f(t, y, \hat{q}) + L_2(\hat{y} - y) \tag{5b}$$
$$\hat{y} = C\hat{x} \tag{5c}$$

where $\hat{x}(t)$ is an **estimate** of the state $x(t)$ of the plant and $\hat{x}(0) = \hat{x}_0$ is an initial estimate of the initial plant state $x_0 = x(0)$. Such an observer is simply a copy of the plant modified with two correction terms: a **Luenberger-type correction term** $L_1(\hat{y} - y)$ characterized by the gain matrix $L_1 \in \mathbb{R}^{n_x \times n_y}$ and an **injection term** $L_2(\hat{y} - y)$ acting on the nonlinearity, characterized by the gain matrix $L_2 \in \mathbb{R}^{n_q \times n_y}$.

Let $e \triangleq \hat{x} - x$ be the **state estimation error** and let

$$\Delta q \triangleq \hat{q} - q.$$

Then, the **observer error dynamics** are described by

$$\dot{e} = (A + L_1 C)e + B_n \Delta f - (B + L_1 D)w \tag{6a}$$
$$\Delta f = f(t, y, q + \Delta q) - f(t, y, q) \tag{6b}$$
$$\Delta q = (C_q + L_2 C)e + D_{qn}\Delta f - (D_{qw} + L_2 D)w \tag{6c}$$

B. *$\mathcal{L}_\infty$-stability with specified performance*

Let

$$z = He \tag{7}$$

be a user-defined **performance output** associated with the state estimation error. Next, we define $\mathcal{L}_\infty$-stability with performance level γ.

Definition 2. *The nonlinear system* (6) *with performance output* (7) *is* ***globally uniformly $\mathcal{L}_\infty$-stable with performance level*** γ *if the following conditions are satisfied.*

(P1) **Global uniform exponential stability.** *The zero-input system (obtained by setting $w \equiv 0$) is globally uniformly exponentially stable about the origin.*

(P2) **Global uniform boundedness of the error state.** *For every initial condition $e(t_0) = e_0$, and every bounded exogenous input $w(\cdot)$, there exists a bound $\beta(e_0, \|w(\cdot)\|_\infty)$ such that*

$$\|e(t)\| \leq \beta(e_0, \|w(\cdot)\|_\infty)$$

for all $t \geq t_0$.

(P3) **Output response for zero initial error state.** *For zero initial error, $e(t_0) = 0$, and every bounded exogeneous input $w(\cdot)$, we have*

$$\|z(t)\| \leq \gamma \|w(\cdot)\|_\infty$$

for all $t \geq t_0$.

(P4) **Global ultimate output response.** *For every initial condition, $e(t_0) = e_0$, and every bounded exogeneous input $w(\cdot)$, we have*

$$\limsup_{t\to\infty} \|z(t)\| \leq \gamma \|w(\cdot)\|_\infty \tag{8}$$

Moreover, convergence is uniform with respect to t_0.

For additional background and definitions, we refer the interested reader to [20].

Our **objective** is to design an observer of the form (5) for the nonlinear system (1) with unknown exogenous inputs whilst ensuring that the observer error dynamics are $\mathcal{L}_\infty$-stable with a specified performance level for a given performance output, described in (7). To this end, the following result is useful.

Lemma 1. *Consider a system with exogenous input w and performance output z described by*

$$\dot{e} = F(t,e,w) \tag{9a}$$
$$z = G(t,e) \tag{9b}$$

where $t \in \mathbb{R}$, $e(t) \in \mathbb{R}^{n_x}$, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$. Suppose there exists a differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ and scalars $\alpha, \beta_1, \beta_2 > 0$ and $\mu_1, \mu_2 \geq 0$ such that

$$\beta_1 \|e\|^2 \leq V(e) \leq \beta_2 \|e\|^2 \tag{10}$$

and

$$\mathcal{D}V(e)\, F(t,e,w) \leq -2\alpha \left(V(e) - \mu_1 \|w\|^2\right) \tag{11a}$$
$$\|G(t,e)\|^2 \leq \mu_2 V(e) \tag{11b}$$

for all $t \geq 0$, $e \in \mathbb{R}^{n_x}$ and $w \in \mathbb{R}^{n_w}$, where $\mathcal{D}V$ denotes the derivative of V. Then system (9) *is globally uniformly $\mathcal{L}_\infty$-stable with performance level*

$$\gamma = \sqrt{\mu_1 \mu_2}. \tag{12}$$

C. Sufficient conditions for observer design with guaranteed performance

We now use the above result to obtain sufficient conditions on the observer gain matrices so that the error system (6) has the desired performance.

Theorem 1. *Consider plant* (1) *and suppose that there are scalars $\alpha > 0$, $\mu \geq 0$, a symmetric matrix $P \succ 0$, matrices L_1, L_2 and an incremental multiplier matrix M for f such that the matrix inequalities*

$$\Phi + \Gamma^\top M \Gamma \preceq 0 \tag{13a}$$

$$\begin{bmatrix} P & \star \\ H & \mu I \end{bmatrix} \succeq 0 \tag{13b}$$

are satisfied where

$$\Phi = \begin{bmatrix} \Phi_{11} & PB_n & -P(B + L_1 D) \\ \star & 0 & 0 \\ \star & 0 & -2\alpha I \end{bmatrix} \tag{14}$$

with

$$\Phi_{11} = P(A + L_1 C) + (A + L_1 C)^\top P + 2\alpha P \tag{15}$$

and

$$\Gamma = \begin{bmatrix} C_q + L_2 C & D_{qn} & -D_{qw} - L_2 D \\ 0 & I & 0 \end{bmatrix}. \tag{16}$$

Then observer (5) *results in error dynamics which are $\mathcal{L}_\infty$-stable with performance level*

$$\gamma = \sqrt{\mu}$$

for the performance output $z = He$.

D. LMI conditions with fixed L_2

The matrix inequality (13b) is linear in the variables P and μ. However matrix inequality (13a) is not an LMI in the variables α, P, M, L_1, L_2 and M. One way to obtain LMI conditions is to fix α and L_2 and introduce a new variable $Y_1 \triangleq PL_1$. Then, inequality (13a) can be rewritten as in (17), which is an LMI in P, Y_1 and M, where Γ is defined in (16). This is summarized in the following corollary of Theorem 1.

Corollary 1. *Consider plant* (1) *and suppose that, for a given scalar $\alpha > 0$ and matrix L_2, there is a scalar $\mu \geq 0$, a symmetric matrix $P \succ 0$, a matrix Y_1 and an incremental multiplier matrix M for f such that the LMI conditions*

$$\Xi + \Gamma^\top M \Gamma \preceq 0 \tag{17}$$

and (13b) *hold, where*

$$\Xi = \begin{bmatrix} \Xi_{11} & PB_n & -PB - Y_1 D \\ \star & 0 & 0 \\ \star & 0 & -2\alpha I \end{bmatrix} \tag{18}$$

with

$$\Xi_{11} = PA + A^\top P + Y_1 C + C^\top Y_1^\top + 2\alpha P, \tag{19}$$

and Γ is defined in (16)*. Then the observer* (5) *with*

$$L_1 = P^{-1} Y_1 \tag{20}$$

has error dynamics which are $\mathcal{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu}$ for output He.

Remark 3. LMI conditions with variable L_2 is provided in [17, Section V] for the interested reader.

V. Estimation of the Exogenous Input

In this section, we consider the problem of estimating the exogeneous input w in addition to the plant state. Using the observers presented in the previous sections, we demonstrate how one can obtain an estimate of w, given that w and its derivative $\dot{w}$ are bounded. We will require that B has full column rank. This implies that it has a left-inverse (for example, the Moore-Penrose pseudoinverse), that is, there is a matrix $B^\dagger$ such that $B^\dagger B = I$. Herein, we will show that

$$\boxed{\hat{w} \triangleq B^\dagger L_1 (\hat{y} - y)} \tag{21}$$

is an estimate of the exogenous input. We discuss exogenous input estimation for two cases based on the structure of the nonlinearity f.

A. Case I: $f = 0$

In this case, the plant is described by

$$\dot{x} = Ax + Bw + g(t,y) \tag{22a}$$
$$y = Cx + Dw \tag{22b}$$

and the proposed observer has the form

$$\dot{\hat{x}} = A\hat{x} + L_1(\hat{y} - y) + g(t,y) \tag{23a}$$
$$\hat{y} = C\hat{x}. \tag{23b}$$

With state estimation error $e = \hat{x} - x$, the error dynamics corresponding to observer (23) are given by

$$\dot{e} = (A + L_1 C)e - (B + L_1 D)w. \tag{24}$$

1) Estimating the exogenous input w: The following theorem provides conditions which, if satisfied, can be used to estimate the exogeneous input to a specific degree of accuracy.

Theorem 2. *Consider plant* (22) *with B full column rank. Suppose there exist scalars $\alpha > 0$, $\mu_1, \mu_2 \geq 0$, a symmetric matrix $P \succ 0$, and a matrix Y_1 that satisfy the following matrix inequalities:*

$$\begin{bmatrix} \Xi_{11} & -PB - Y_1 D \\ \star & -2\alpha I \end{bmatrix} \preceq 0 \tag{25a}$$

$$\begin{bmatrix} P & \star \\ B^\dagger A & \mu_1 I \end{bmatrix} \succeq 0 \tag{25b}$$

$$\begin{bmatrix} P & \star \\ B^\dagger & \mu_2 I \end{bmatrix} \succeq 0 \tag{25c}$$

where Ξ_{11} is given by (19). *Then observer* (23) *with*

$$L_1 = Y_1 P^{-1} \tag{26}$$

and $\hat{w}$ given by (21) *yields the exogenous input estimation error bound*

$$\limsup_{t\to\infty} \|\hat{w}(t) - w(t)\| \leq \sqrt{\mu}_1 \|w(\cdot)\|_\infty + \sqrt{\mu}_2 \|\dot{w}(\cdot)\|_\infty. \tag{27}$$

Proof: Recalling the plant description (22) and the observer description (23), we see that the error dynamics can be described by

$$\dot{e} = Ae + L_1(\hat{y} - y) - Bw. \tag{28}$$

Pre-multiplying this equation by $B^\dagger$ and recalling definition (21) of $\hat{w}$ results in

$$B^\dagger \dot{e} = B^\dagger A e + \hat{w} - w\,.$$

Hence $\hat{w} - w = B^\dagger \dot{e} - B^\dagger A e$ which implies that

$$\|\hat{w} - w\| \leq \|B^\dagger \dot{e}\| + \|B^\dagger A e\|\,. \tag{29}$$

We now use Corollary 1 to obtain ultimate bounds on $\|B^\dagger \dot{e}\|$ and $\|B^\dagger A e\|$. To this end, we first note that inequality (17) reduces to (25a) when B_n and M are zero matrices.

It now follows from Corollary 1 that satisfaction of (25a) and (25b) implies that the error system (24) with performance output $B^\dagger A e$ is $\mathcal{L}_\infty$-stable with performance level $\sqrt{\mu_1}$. Hence the ultimate bound on $B^\dagger A e$ satisfies

$$\limsup_{t\to\infty} \|B^\dagger A e(t)\| \leq \sqrt{\mu_1} \|w(\cdot)\|_\infty\,. \tag{30}$$

Computing the time-derivative of the error system (24) we obtain

$$\ddot{e} = (A + L_1 C)\dot{e} - (B + L_1 D)\dot{w}\,. \tag{31}$$

Considering this as a system with state $\dot{e}$, exogenous input $\dot{w}$, and performance output $B^\dagger \dot{e}$, it follows from Corollary 1 that satisfaction of (25a) and (25c) implies $\mathcal{L}_\infty$-stability with performance level $\sqrt{\mu_2}$. Hence the ultimate bound on $B^\dagger \dot{e}(t)$ satisfies

$$\limsup_{t\to\infty} \|B^\dagger \dot{e}(t)\| \leq \sqrt{\mu_2} \|\dot{w}(\cdot)\|_\infty\,. \tag{32}$$

Combining (29), (30) and (32) yields

$$\begin{aligned} \limsup_{t\to\infty} \|\hat{w}(t) - w(t)\| &\leq \limsup_{t\to\infty} \Big(\|B^\dagger \dot{e}(t)\| + \|B^\dagger A e(t)\| \Big) \\ &\leq \limsup_{t\to\infty} \|B^\dagger \dot{e}(t)\| + \limsup_{t\to\infty} \|B^\dagger A e(t)\| \\ &\leq \sqrt{\mu_1} \|w(\cdot)\|_\infty + \sqrt{\mu_2} \|\dot{w}(\cdot)\|_\infty\,, \end{aligned}$$

which is the desired upper bound (27). ■

2) Estimating w to an arbitrary degree of accuracy: The following result is an immediate consequence of Therorem 3.

Corollary 2. *Consider plant* (22) *with B full column rank and suppose that for every $\mu > 0$ there there exist a scalar $\alpha > 0$, a symmetric matrix $P \succ 0$, and a matrix Y_1 that satisfy* (25) *with $\mu_1 = \mu_2 = \mu$. Let w be a bounded exogenous input with bounded derivative. Then, for any $\varepsilon > 0$, there is an observer of the form* (23) *with L_1 given by* (26) *so that*

$$\limsup_{t\to\infty} \|\hat{w}(t) - w(t)\| \leq \varepsilon, \tag{33}$$

where $\hat{w}$ is given by (21).

Corollary 2 provides conditions for the estimation of the unknown exogenous input w with arbitrary accuracy. We will illustrate in the following theorem that Corollary 2 is a generalized result of the conditions previously reported in the construction of unknown input observers for linear systems satisfying the so-called 'matching condition'—see for example: [21]–[23].

Theorem 3. *Consider plant* (22) *with $D = 0$ and B full column rank. Suppose that there is a symmetric matrix $\tilde{P} \succ 0$, and matrices $\tilde{Y}$ and $\tilde{F}$ such that*

$$\tilde{P}A + A^\top \tilde{P} + \tilde{Y}C + C^\top \tilde{Y}^\top \prec 0 \tag{34a}$$

$$B^\top \tilde{P} - \tilde{F}C = 0 \tag{34b}$$

Let w be a bounded exogenous input with bounded derivative. Then, for any $\varepsilon > 0$, there is an observer of the form (23) *so that* (33) *holds where $\hat{w}$ is given by* (21).

Proof: We will prove this by showing that the hypotheses of Corollary 2 are satisfied. Suppose $\tilde{P}, \tilde{Y}$ and $\tilde{F}$ satisfy (34) with $\tilde{P}^\top = \tilde{P} \succ 0$. Consider any $\mu > 0$. Since $\tilde{P} \succ 0$, there exists a scalar $\nu > 0$ such that

$$P \succeq \mu^{-1}(B^\dagger A)^\top B^\dagger A \quad \text{and} \quad P \succeq \mu^{-1}(B^\dagger)^\top B^\dagger \tag{35}$$

where

$$P := \nu \tilde{P}. \tag{36}$$

Satisfaction of (35) implies satisfaction of (25b) and (25c). Let $Y = \nu \tilde{Y}$ and $F = \nu \tilde{F}$. Then satisfaction of (34a) by $\tilde{P}$ and $\tilde{Y}$ implies there exists $\alpha > 0$ such that

$$PA + A^\top P + YC + C^\top Y^\top + 2\alpha P \preceq 0. \tag{37}$$

Also, (34b) implies that

$$B^\top P = FC. \tag{38}$$

Choosing

$$\zeta \geq \|F\|^2/4\alpha, \tag{39}$$

we have $F^\top F \leq 4\alpha\zeta I$; hence $\frac{1}{2\alpha}C^\top F^\top FC - 2\zeta C^\top C \preceq 0$. Using (38), we get

$$\frac{1}{2\alpha}PBB^\top P - 2\zeta C^\top C \preceq 0. \tag{40}$$

Combing (37) and (40) and letting

$$Y_1 = Y - \zeta C \tag{41}$$

results in

$$PA + A^\top P + Y_1 C + C^\top Y_1^\top + 2\alpha P + \frac{1}{2\alpha}PBB^\top P \preceq 0. \tag{42}$$

Since $\alpha > 0$ and $D = 0$, using Schur complements, the above inequality is equivalent to (25a). ■

B. Case II: $f \neq 0$

For simplicity, we consider nonlinear functions with q as their only argument. This yields a plant of the form

$$\dot{x} = Ax + B_n f(q) + Bw + g(t, y) \tag{43a}$$
$$q = C_q x + D_{qn} f(q) + D_{qw} w \tag{43b}$$
$$y = Cx + Dw. \tag{43c}$$

For such plants the proposed observers are described by

$$\dot{\hat{x}} = A\hat{x} + B_n f(\hat{q}) + L_1(\hat{y} - y) + g(t, y) \tag{44a}$$
$$\hat{q} = C_q \hat{x} + D_{qn} f(\hat{q}) + L_2(\hat{y} - y) \tag{44b}$$
$$\hat{y} = C\hat{x}. \tag{44c}$$

We make the following additional assumptions for the class of systems considered in this subsection.

Assumption 2. *The function f is differentiable and there is a scalar κ_1 such that $\|\mathcal{D}f(q)\| \leq \kappa_1$ for all $q \in \mathbb{R}^{n_q}$ and $\kappa_1\|D_{qn}\| < 1$.*

Assumption 3. *The derivative $\dot{x}$ of the state of plant (43) is bounded.*

Remark 4. Assumption 2 implies that the nonlinearity f is globally κ_1-Lipschitz, that is, $\|f(\hat{q}) - f(q)\| \leq \kappa_1\|\hat{q} - q\|$ for all $\hat{q}, q \in \mathbb{R}^{n_q}$. This assumption also guarantees that there exists a scalar κ_2 such that $\|\mathcal{D}f(\hat{q}) - \mathcal{D}f(q)\| \leq \kappa_2$ for all $\hat{q}, q \in \mathbb{R}^{n_q}$.

1) Estimating the exogenous input w: We now state and prove the following theorem that provides sufficiency conditions for the estimation of the exogeneous input w to a specified degree of accuracy.

Theorem 4. *Consider plant (43) with B full column rank and satisfying Assumptions 2 and 3. Suppose there exist scalars $\alpha > 0$, $\mu_1, \mu_2, \mu_3 \geq 0$, a symmetric matrix $P \succ 0$, matrices L_1 and L_2 and an incremental multiplier matrix M for f, such that*

$$\hat{\Phi} + \hat{\Gamma}^\top M \hat{\Gamma} \preceq 0 \tag{45a}$$

$$\begin{bmatrix} P & \star \\ B^\dagger A & \mu_1 I \end{bmatrix} \succeq 0 \tag{45b}$$

$$\begin{bmatrix} P & \star \\ C_q + L_2 C & \mu_2 I \end{bmatrix} \succeq 0 \tag{45c}$$

$$\begin{bmatrix} P & \star \\ B^\dagger & \mu_3 I \end{bmatrix} \succeq 0 \tag{45d}$$

where

$$\hat{\Phi} = \begin{bmatrix} \Phi_{11} & PB_n & -P(B + L_1 D) & PB_n \\ \star & 0 & 0 & 0 \\ \star & 0 & -2\alpha I & 0 \\ \star & 0 & 0 & -2\alpha I \end{bmatrix}$$

Φ_{11} is given by (15) and

$$\hat{\Gamma} = \begin{bmatrix} C_q + L_2 C & D_{qn} & -D_{qw} - L_2 D & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Then the observer (44) with $\hat{w}$ given by (21) yields the exogenous input estimation error bound:

$$\limsup_{t\to\infty} \|\hat{w} - w\| \leq \gamma_1\|w(\cdot)\|_\infty + \gamma_2\|\dot{w}(\cdot)\|_\infty + \gamma_3\|\dot{x}(\cdot)\|_\infty,$$

where

$$\gamma_1 = \sqrt{\mu_1} + \tilde{\kappa}_1\|B^\dagger B_n\|(\sqrt{\mu_2} + \|D_{qw} + L_2 D\|)$$
$$\gamma_2 = \sqrt{\mu_3}(1 + \tilde{\kappa}_2\|D_{qw}\|)$$
$$\gamma_3 = \sqrt{\mu_3}\tilde{\kappa}_2\|C_q\|$$
$$\tilde{\kappa}_1 = \kappa_1(1 - \kappa_1\|D_{qn}\|)^{-1}$$
$$\tilde{\kappa}_2 = \kappa_2(1 - \kappa_1\|D_{qn}\|)^{-1}.$$

2) Estimating w to an arbitrary degree of accuracy: The following result is an immediate consequence of Therorem 3.

Corollary 3. *Consider plant (43) with B full column rank and satisfying Assumptions 2 and 3. Suppose that, for every $\mu > 0$ there is a scalar $\alpha > 0$, a symmetric matrix $P \succ 0$, matrices L_1 and L_2 and an incremental multiplier matrix M for f that satisfy (45) with $\mu_1 = \mu_2 = \mu_3 = \mu$ and $D_{qw} + L_2 D = 0$. Let w be a bounded exogenous input with bounded derivative. Then, for any $\varepsilon > 0$, there exists an observer of the form (44) that satisfies*

$$\limsup_{t\to\infty} \|\hat{w}(t) - w(t)\| \leq \varepsilon,$$

where $\hat{w}$ is given by (21).

VI. Example

In this section, we test the performance of the proposed observer on the following nonlinear system with disturbance w.

$$\dot{x} = \begin{bmatrix} -x_2 + w \\ -x_2 - x_3 + (x_1 - x_2)^3 \\ x_2 - x_3 \end{bmatrix}, \quad y = x_1 \tag{47}$$

to illustrate asymptotic estimation of the unknown input signal w. We rewrite the model (47) in the form (1) with

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad B_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C_q = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \qquad D_{qn} = D_{qw} = 0$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad D = 0$$

$g = 0$ and $f(t, y, q) = q^3$. Since the nonlinearity f is incrementally positive real, incremental multiplier matrices are given by

$$M = \kappa \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for any $\kappa > 0$ (see Example 1). We choose $H = I_3$, and fix our exponential decay rate $\alpha = 1$. Solving (45) with $L_2 = 0$ we get

$$P = \begin{bmatrix} 3915.89 & 0.13 & 0 \\ 0.13 & 7.15 & 0 \\ 0 & 0 & 7.15 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -4.3 \times 10^4 \\ -0.1076 \\ 0.9771 \end{bmatrix}^\top,$$

$\kappa = 7.1506$, and $\gamma = 3.8 \times 10^{-4}$. As the magnitude of γ is small, we expect to reconstruct the unknown input signal w. We test our proposed observer on the system (47) with the initial conditions $x(0) = \begin{bmatrix} 426.39 & -372.81 & -236.45 \end{bmatrix}^\top$ and $\hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. The response of the proposed observer is shown in Figure 1. We observe that the observation error becomes arbitrarily small and the unknown input is estimated to satisfactory accuracy.

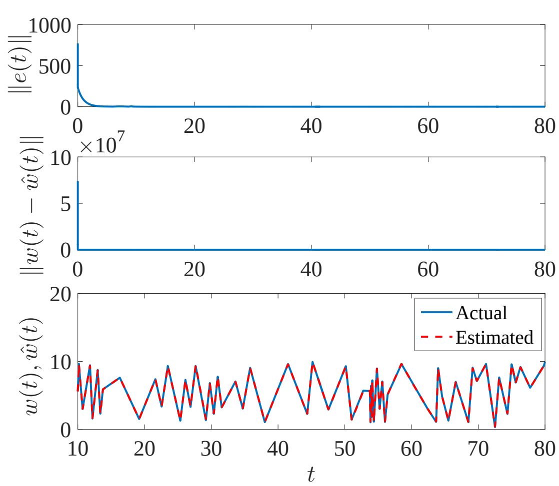

Fig. 1. (*Top*) State estimation error of the nonlinear system in (47). (*Middle*) Unknown Input Estimation Error. The convergence of $\|w(t) - \hat{w}(t)\|$ is illustrated. (*Bottom*) Unknown Input Estimation. The simulated unknown input $w(t)$ is shown in blue and the reconstruction $\hat{w}(t)$ is depicted with red.

VII. Conclusions

In this paper, we develop a method for constructing state observers for a general class of nonlinear systems with unknown but bounded exogenous inputs. The proposed observers can be applied to biomedical systems, networked systems, intelligent systems and mechanical systems. Furthermore, the convexity of the design make the proposed method attractive and reliable from a computational viewpoint.

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